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Application of a New Hybrid Method for Solving System of Nonlinear BVPs Arising in Fluid Mechanics

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ABSTRACT

This article is aimed to introduce a new hybrid analytical-digital technique for solving a wide range of problems in fluid mechanics. This method is according to the Different Transform Method (DTM) and Newton's Iterative Method (NIM). In the Boundary Value Problems (BVP), the system and the boundary conditions converted to an algebraic equation set, and the Taylor series of the solution are subsequently calculated. By finding Jacobian matrix, the unknown parameters of the solution may be calculated using the multi-variable iterative Newton's method. The techniques are employed to determine a proximate solution for the problem. To expound upon the application of the new hybrid method illustratively, two nonlinear problems in fluid mechanics are considered: condensation film on the inclined rotating disk and the rotating MHD flow on a porous shrinking sheet. Using comparing the present results obtained with the numerical solutions and results presented in the literature, an excellent accuracy is observed. Quick convergence of the solution is another important merit of the proposed method.

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1. Introduction

Generally, the scientific problems and phenomena are necessarily nonlinear all over the world and may be modeled with the help of nonlinear differential equations. The majority of these problems do not have an exact analytical solution. So the scientists use the numerical solution and proximate procedures for solving such kind of equations. Obtaining a complete form of results through the numerical solution is expensive and time-consuming since the solution is given at the discrete points. Moreover, the stability and convergence must be taken into account in the numerical solution to avoid divergence or inappropriate results.

Proximate methods such as Homotopy Perturbation Method (HPM), decomposition method (DM), Variational Iteration Method (VIM) and homotopy analysis method (HAM) are good substitutes for the numerical ones. Recently, some of the nonlinear engineering problems have been solved through some of the methods such as HAM [1–9], HPM [10–14], VIM [15–17] and DM [18–21]. In most studies, some modifications are introduced to eliminate the complexity and nonlinearity of the problems.

Differential transform method (DTM) is another approximate method for solving differential equations. This method uses an iterative technique for finding the Taylor series solution for the problem. According to this method, it is not necessary to pay for high calculation costs to obtain the Taylor series coefficients. Zhou introduced the DTM for solving the initial value problems in the electrical circuit analysis. Then, the DTM was applied on differential algebraic equations [22,23], partial differential equations [24–29], integral equations [30–32], ordinary differential equations [33–37] and fractional differential equations [38–41].

Here, we report a short review about the previous works in the literature about the nonlinear ordinary differential equation solution. This method may be applicable on the initial value problems straightforwardly, because it works based on the initial conditions. There are some initial value problems that have been solved using the classic DTM in [42–44].Usually the solution obtained by DTM will diverge for large domain, because we should limit the number of Taylor series elements. To overcome such a problem, two techniques may be used: multi-stepDTM [45–47] and after treatment Pade approximation [48–51].

Solution of boundary value problems is a main application of DTM in engineering problems. To solve this kind of problems, one of the boundary conditions transforms to an unknown initial condition. Then, the solution can be obtained as a function of the unknown parameter. Finally, by applying the boundary condition transformed to initial condition, the value of the unknown parameter can be computed. This technique first is used in [33] to solve the steady thermal transfer equations. Later, some of the nonlinear thermal transfer equations of fins are solved by DTM such as [52–55].

In this article, we use a combination of differential transform method and iterative Newton's method (INM) for solving the system of high order nonlinear boundary value problems. The unknown parameters are determined by the INM as a shooting method. As we will see in the results, the solution converges rapidly with 5 to 7 iterations. We chose two nonlinear problems subject to the fluid mechanics and thermal transfer to illustrate how we can apply the new hybrid method on the problems.

2. Differential transform method

The differential transform been defined as it follows:

$$X(k)=\frac{1}{k!}\left[\frac{d^k x(t)}{dt^k}\right]_{t=t_0}. \tag{1}$$

Where *x(t)* indicates an arbitrary function, and *X(k)* indicates the transformed function. The inverse transformation is as follows

$$x(t)=\sum_{k=0}^{\infty}X(k)(t-t_0)^k. \tag{2}$$

By substituting Eq(1) in Eq (2)

$$x(t)=\sum_{k=0}^{\infty}\frac{(t-t_0)^k}{k!}\left[\frac{d^k x(t)}{dt^k}\right]_{t=t_0}. \tag{3}$$

The function *x(t)* is often considered as a series with limited terms and Eq. (2), may be rewritten as:

The *x(t)*function is often considered as a series having limited terms and Eq(2)

$$x(t)\approx\sum_{k=0}^{m}X(k)(t-t_0)^k. \tag{4}$$

where, *m* is the number of Taylor series' components. Generally, the accuracy of the solution can be increased by raising this value

Some of the DTM properties are presented in table1. They are obtained from Eqs. (1) and (2).

Table 1
The properties of the DTM.

Original function	Transformed function
$f(t)=g(t)\pm h(t)$	$F(k)=G(k)\pm H(k)$
$f(t)=cg(t)$	$F(k)=cG(k)$
$f(t)=\frac{d^n g(t)}{dt^n}$	$F(k)=\frac{(k+n)!}{k!}G(k+n)$
$f(t)=g(t)h(t)$	$F(k)=\sum_{r=0}^{k}G(r)H(k-r)$
$f(t)=t^n$	$F(k)=\delta(k-n)=\begin{cases}1 & if\ k=n\\ 0 & if\ k\neq n\end{cases}$

3. Applications

3.1. Condensation film on the inclined rotating disk

3.1.1. Mathematical formulation

The dimensionless momentum and energy equations of the condensation film on the inclined rotating disk may be formulated as the nonlinear system of boundary value problems as follow [56]:

$$f''' - (f')^2 + g^2 + 2ff'' = 0, \tag{5}$$

$$g'' - 2gf' + 2fg' = 0, \tag{6}$$

$$p'' - pf' + sg + 2fp' + 1 = 0, \tag{7}$$

$$s'' - gp - sf' + 2fs' = 0, \tag{8}$$

$$\theta'' + 2\,\mathrm{Pr} f\,\theta' = 0. \tag{9}$$

Regarding the boundary conditions:

$$\begin{aligned} &f(0)=0, \quad f'(0)=0, \quad f''(\delta)=0,\\ &g(0)=1, \quad g'(\delta)=0,\\ &p(0)=0, \quad p'(\delta)=0,\\ &s(0)=0, \quad s'(\delta)=0,\\ &\theta(0)=0, \quad \theta(\delta)=1. \end{aligned} \tag{10}$$

Where Pr and δ are the Prandtle number and dimensionless thickness, respectively.

3.1.2. Solution with the new hybrid method

In this section, Eqs 5 to 9 are solved by applying a new hybrid technique. The solution includes two steps. First the Taylor series of solution is determined by using the mathematical relations and applying DTM. Finally, the iterative newton's method is used to obtain the unknown parameters.

3.1.2.1. Applying DTM

The boundary value problems (Eqs. (5) to (9)) may be transformed to the initial value ones replacing the unknown initial conditions in place of the boundary conditions at the end.

$$f''(0) = a_1, \quad g'(0) = a_2, \quad p'(0) = a_3, \quad s'(0) = a_4, \quad \theta'(0) = a_5. \tag{11}$$

Applying the DTM on Eqs. (5) to (9) at $\eta = 0$, the recursive relations are determined to calculate the coefficients of the series solutions as follow:

$$F(k+3)=\frac{1}{(k+1)(k+2)(k+3)}\left\{\sum_{r=0}^{k}(r+1)F(r+1)(k-r+1)F(k-r+1)\right.$$
$$\left.-\sum_{r=0}^{k}G(r)G(k-r)-2\sum_{r=0}^{k}(r+1)(r+2)F(r+2)F(k-r)\right\}, \tag{12}$$

$$G(k+2)=\frac{2}{(k+1)(k+2)}\left\{\sum_{r=0}^{k}(r+1)F(r+1)G(k-r)-\sum_{r=0}^{k}(r+1)G(r+1)F(k-r)\right\}, \tag{13}$$

$$P(k+2)=\frac{1}{(k+1)(k+2)}\left\{\sum_{r=0}^{k}(r+1)F(r+1)P(k-r)\right.$$
$$\left.-\sum_{r=0}^{k}S(r)G(k-r)-2\sum_{r=0}^{k}(r+1)P(r+1)F(k-r)-\delta(k)\right\}, \tag{14}$$

$$S(k+2)=\frac{1}{(k+1)(k+2)}\left\{\sum_{r=0}^{k}G(r)P(k-r)\right.$$
$$\left.+\sum_{r=0}^{k}(r+1)F(r+1)S(k-r)-2\sum_{r=0}^{k}(r+1)S(r+1)F(k-r)\right\}, \tag{15}$$

$$\Theta(k+2)=\frac{1}{(k+1)(k+2)}\left\{-2\Pr\sum_{r=0}^{k}(r+1)\Theta(r+1)F(k-r)\right\}, \tag{16}$$

The differential transform of the conditions at $\eta=0$ in Eq. (11) is:

$$\begin{aligned}
&F(0)=0, \quad G(0)=1, \quad P(0)=0, \quad S(0)=0, \quad \Theta(0)=0,\\
&F(1)=0, \quad G(1)=a_2, \quad P(1)=a_3, \quad S(1)=a_4, \quad \Theta(1)=a_5,\\
&F(2)=a_1.
\end{aligned} \tag{17}$$

Substituting Eq. (17) into the Eqs. (12) to (16) for *k=0,1,…* , we have:

$$f(\eta)=\frac{a_1}{2}\eta^2-\frac{1}{6}\eta^3-\frac{1}{12}a_2\eta^4-\frac{1}{60}a_2^2\eta^5-\frac{1}{360}a_1\eta^6+\left(\frac{1}{2520}-\frac{1}{630}a_2a_1\right)\eta^7+\ldots, \tag{18}$$

$$g(\eta)=1+a_2\eta-\frac{1}{3}a_1\eta^3+\left(\frac{1}{12}a_1a_2-\frac{1}{12}\right)\eta^4-\frac{1}{15}a_2\eta^5-\left(\frac{1}{90}a_1^2+\frac{1}{45}a_2^2\right)\eta^6+\ldots, \tag{19}$$

$$p(\eta)=a_3\eta-\frac{1}{2}\eta^2-\frac{1}{6}a_4\eta^3-\frac{1}{12}a_2a_4\eta^4+\left(\frac{1}{40}a_1-\frac{1}{60}a_3\right)\eta^5-\left(\frac{1}{720}+\frac{1}{72}a_2a_3\right)\eta^6+\ldots, \tag{20}$$

$$s(\eta)=a_4\eta+\frac{1}{6}a_3\eta^3+\left(\frac{1}{12}a_2a_3-\frac{1}{24}\right)\eta^4-\left(\frac{1}{60}a_4+\frac{1}{40}a_2\right)\eta^5-\frac{1}{72}a_2a_4\eta^6+\ldots, \tag{21}$$

$$\theta(\eta)=a_5\eta-\frac{1}{1200}a_1a_5\eta^4+\frac{1}{6000}a_5\eta^5+\frac{1}{18000}a_2a_5\eta^6+\frac{1}{126000}\left(a_1^2+a_2^2\right)a_5\eta^7+\ldots, \tag{22}$$

3.1.2.2. Applying the iterative newton's method

In this part, the unknown parameters are obtained from the boundary conditions and $\eta=\delta$ is substituted in Eqs 18 to 22. With respect to this issue the following residual functions are defined ro minimize them to be able to obtain the unknown parameters.

$$\begin{aligned}
&R_1=f''(\delta,a_1,a_2,a_3,a_4,a_5)=\sum_{k=2}^{m}k(k-1)F(k)\delta^{k-2},\\
&R_2=g'(\delta,a_1,a_2,a_3,a_4,a_5)=\sum_{k=1}^{m}kG(k)\delta^{k-1},\\
&R_3=p'(\delta,a_1,a_2,a_3,a_4,a_5)=\sum_{k=1}^{m}kP(k)\delta^{k-1},\\
&R_4=s'(\delta,a_1,a_2,a_3,a_4,a_5)=\sum_{k=1}^{m}kS(k)\delta^{k-1},\\
&R_5=\theta(\delta,a_1,a_2,a_3,a_4,a_5)-1=\sum_{k=0}^{m}\Theta(k)\delta^{k}-1.
\end{aligned}\tag{23}$$

To obtain the values of a_1 to a_5 the aforesaid functions must be zero. For obtaining the roots of the Eq. (23), we apply the multi-variable iterative method of Newoton as follows:

$$\begin{bmatrix}a_1\\a_2\\a_3\\a_4\\a_5\end{bmatrix}_{n+1}=\begin{bmatrix}a_1\\a_2\\a_3\\a_4\\a_5\end{bmatrix}_{n}-\begin{bmatrix}\frac{\partial R_1}{\partial a_1}&\cdots&\frac{\partial R_1}{\partial a_5}\\ \vdots&\ddots&\vdots\\ \frac{\partial R_5}{\partial a_1}&\cdots&\frac{\partial R_5}{\partial a_5}\end{bmatrix}_{n}^{-1}\begin{bmatrix}R_1\\R_2\\R_3\\R_4\\R_5\end{bmatrix}_{n},\qquad n=0,1,2,\ldots\tag{24}$$

By surmising the initial values of a_1 to a_5, we should compute the residual vector (R) and Jacobian Matrix ($\frac{\partial R_i}{\partial a_j}$). The residual vector may be obtained through substituting $(a_1,\ldots,a_5)^n$ in Eq. (23). The components of the Jacobian matrix in Eq. (24) may be calculated through differentiating them analytically with respect to a_1 to a_5 and after that, substituting $(a_1,\ldots,a_5)^n$ in that equation.

The efficiency and accuracy of the proposed hybrid method is indicated by illustrating the results and numerical solution diagrams in figures 1 and 2. The present results are compared with the numerical solution using the Runge-Kutta method in these figures. The proximate solution of the problem is shown in Table 2 for *Pr*=1. The values of the unknown parameters a_1,a_2,a_3,a_4 and a_5 shown in Table 3 for *Pr*=5 and also different numbers of thickness. These values may be substituted in Eqs. (18) to (22) to determine the approximate solution of the problem.

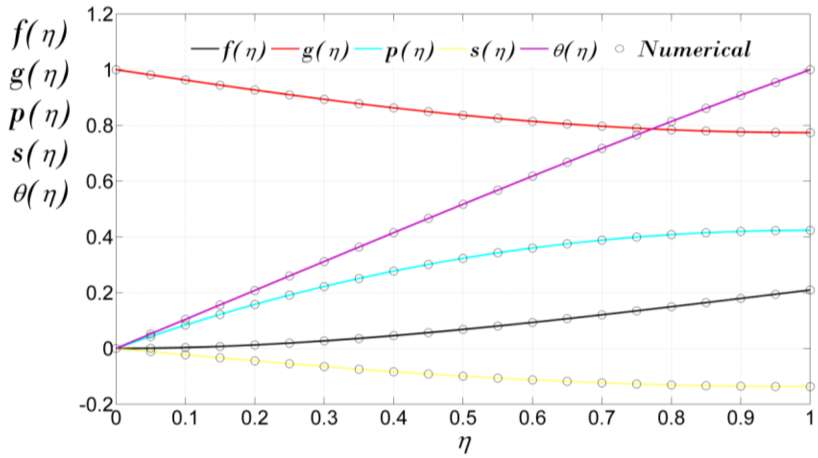


Fig. 1. The profiles $f(\eta), g(\eta), p(\eta), s(\eta)$ and $\theta(\eta)$ when $\delta = 1$ and Pr=1.

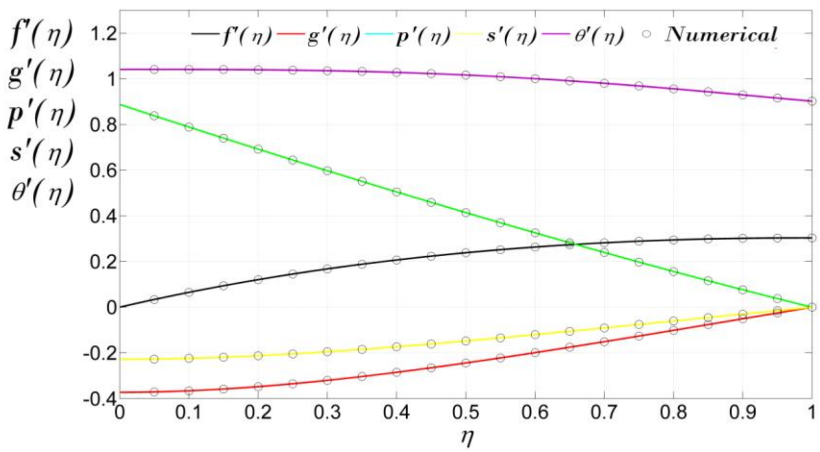


Fig. 2. The profiles $f'(\eta), g'(\eta), p'(\eta), s'(\eta)$ and $\theta'(\eta)$ when $\delta = 1$ and Pr=1.

Table 2
The proximate solutions $f(\eta), g(\eta), p(\eta), s(\eta)$ and $\theta(\eta)$ when Pr=1.

	Approximate solution
	$f(\eta) = 0.2412\eta^2 - 0.1667\eta^3 + 0.0066\eta^4 - 0.0001\eta^5 - 0.0013\eta^6 + 0.0004\eta^7 - 0.0000\eta^8$
	$g(\eta) = 1 - 0.0787\eta + 0.1608\eta^3 - 0.0865\eta^4 + 0.0052\eta^5 - 0.0027\eta^6 + 0.0016\eta^7 - 0.0004\eta^8$
$\delta = 0.5$	$s(\eta) = -0.0401\eta + 0.0823\eta^3 - 0.0449\eta^4 + 0.0026\eta^5 - 0.0000\eta^6 - 0.0003\eta^7 + 0.0000\eta^8$
	$p(\eta) = 0.4941\eta - 0.5\eta^2 + 0.0067\eta^3 - 0.0002\eta^4 + 0.0038\eta^5 - 0.0008\eta^6 - 0.0001\eta^7 - 0.0004\eta^8$
	$\theta(\eta) = 2.0080\eta - 0.0807\eta^4 + 0.0335\eta^5 - 0.0009\eta^6 + 0.0037\eta^7 - 0.0033\eta^8$
	$f(\eta) = 0.3489\eta^2 - 0.1667\eta^3 + 0.0311\eta^4 - 0.0023\eta^5 - 0.0019\eta^6 + 0.0008\eta^7 - 0.0001\eta^8$
	$g(\eta) = 1 - 0.3720\eta + 0.2355\eta^3 - 0.1052\eta^4 + 0.0248\eta^5 - 0.0086\eta^6 + 0.0031\eta^7 - 0.0009\eta^8$
$\delta = 1$	$p(\eta) = 0.8933\eta - 0.5\eta^2 + 0.0380\eta^3 - 0.0071\eta^4 + 0.0028\eta^5 + 0.0032\eta^6 - 0.0008\eta^7 - 0.0008\eta^8$
	$s(\eta) = -0.2281\eta + 0.1489\eta^3 - 0.0693\eta^4 + 0.0131\eta^5 - 0.0012\eta^6 + 0.0004\eta^7 - 0.0003\eta^8$
	$\theta(\eta) = 1.0445\eta - 0.0615\eta^4 + 0.0174\eta^5 - 0.0022\eta^6 + 0.0042\eta^7 - 0.0025\eta^8$

Table 3
The values of a_1, a_2, a_3, a_4 and a_5 obtained using the iterative Newton's method when *Pr*=5.

	a_1	a_2	a_3	a_4	a_5
$\delta = 0.1$	0.0999	-0.0006	0.0999	-0.0003	10.0003
$\delta = 0.2$	0.1998	-0.0053	0.1999	-0.0027	5.0026
$\delta = 0.3$	0.2986	-0.0178	0.2995	-0.0089	3.3422
$\delta = 0.4$	0.3940	-0.0416	0.3980	-0.0210	2.5209
$\delta = 0.5$	0.4825	-0.0787	0.4941	-0.0401	2.0399
$\delta = 0.6$	0.5594	-0.1285	0.5862	-0.0667	1.7329
$\delta = 0.7$	0.6208	-0.1879	0.6726	-0.1003	1.5279
$\delta = 0.8$	0.6647	-0.2519	0.7522	-0.1395	1.3885
$\delta = 0.9$	0.6921	-0.3149	0.8249	-0.1823	1.2956
$\delta = 1$	0.7057	-0.3730	0.8910	-0.2272	1.2423

All the initial guesses for a_1 to a_5 are considered at first. The convergence history of a special case is illustrated in Fig3.as it can be seen, the problem converged quickly with only 5 iterations.

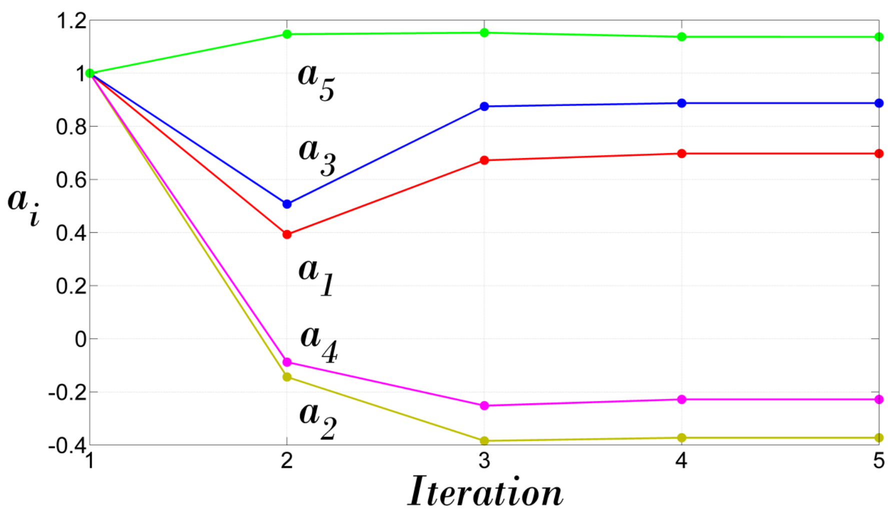


Fig. 3. The history of the iterative method of Newton when $\delta = 1$ and *Pr*=5.

3.2 Rotating MHD flow over a porous shrinking sheet

3.2.1. Mathematical formulation

The dimensionless momentum equations of the Rotating MHD Flow over a Porous Shrinking Sheet may be formulated as the following nonlinear system of boundary value problems [57]:

$$f^{IV} - M^2 f'' - 2K_p^2 g' - \mathrm{Re}\left(f f'' - f f'''\right) = 0, \quad (25)$$

$$g'' - M^2 g + 2K_p^2 f' - \mathrm{Re}\left(f \acute{g} - f g'\right) = 0, \quad (26)$$

with the boundary conditions:

$$\begin{aligned} &f(-1)=\lambda, \quad f'(-1)=-1, \quad g(-1)=0, \\ &f(1)=0, \quad f'(1)=0, \quad g(1)=0, \end{aligned} \tag{27}$$

where M, K_p, Re and λ are the constants of problem that have been introduced in [57].

3.2.2. Solution with the new hybrid method

In this section, Eqs 25 and 26 are solved using a new hybrid technique. This solution is comprised of two stages. Fisrt the Taylor series solution is obtained by the mathematical relations and applying DTM. Then, the unknown parameters are determined via the iterative method of Newton.

3.2.2.1. Applying DTM

The solution to this problem is considered as the Taylor series at $\eta = 0$ in this form as follows:

$$\begin{aligned} &f(\eta)=\sum_{k=0}^{m} F(k)\eta^k, \qquad -1 \le \eta \le 1 \\ &g(\eta)=\sum_{k=0}^{m} G(k)\eta^k, \qquad -1 \le \eta \le 1 \end{aligned} \tag{28}$$

The boundary value problems (Eqs. (25) and (26)) may be transformed to the initial value ones replacing the unknown initial conditions in place of the boundary conditions.

$$\begin{aligned} &f(0)=a_1, \quad f'(0)=a_2, \quad f''(0)=a_3, \quad f'''(0)=a_4, \\ &g(0)=a_5, \quad g'(0)=a_6. \end{aligned} \tag{29}$$

where a_1 to a_6 are the unknown parameters. By applying the DTM on Eqs. (25) and (26) at $\eta = 0$, the recursive relations, as follow, are determined to compute the coefficients of the series solutions

$$\begin{aligned} F(k+4)=&\frac{1}{(k+1)(k+2)(k+3)(k+4)}\Big\{M^2(k+1)(k+2)F(k+2) \\ &+2K_p^2(k+1)G(k+1)+\mathrm{Re}\sum_{r=0}^{k}(r+1)(r+2)F(r+2)(k-r+1)F(k-r+1) \\ &-\mathrm{Re}\sum_{r=0}^{k}(r+1)(r+2)(r+3)F(r+3)F(k-r)\Big\}, \end{aligned} \tag{30}$$

$$\begin{aligned} G(k+2)=&\frac{1}{(k+1)(k+2)}\Big\{M^2G(k)-2K_p^2(k+1)F(k+1) \\ &+\mathrm{Re}\sum_{r=0}^{k}(r+1)F(r+1)G(k-r)-\mathrm{Re}\sum_{r=0}^{k}(r+1)G(r+1)F(k-r)\Big\}. \end{aligned} \tag{31}$$

The differential transform of the conditions in Eq. (27) is as follows:

$$F(0)=a_1,\quad F(1)=a_2,\quad F(2)=\frac{a_3}{2},\quad F(3)=\frac{a_4}{6},$$
$$G(0)=a_5,\quad G(1)=a_6. \tag{32}$$

Substituting Eq. (32) into the Eqs. (30) and (31) for K_p=1, M=1and Re=1, we have:

$$F(4)=\frac{1}{24}a_3+\frac{1}{12}a_6+\frac{1}{24}a_3a_2-\frac{1}{24}a_4a_1,$$
$$F(5)=\frac{1}{120}a_4+\frac{1}{60}a_5-\frac{1}{30}a_2+\frac{1}{60}a_2a_5-\frac{1}{60}a_6a_1+\frac{1}{120}a_3^2-\frac{1}{120}a_1\left(a_3+2a_6+a_3a_2-a_1a_4\right), \tag{33}$$
$$\ldots$$

$$G(2)=\frac{1}{2}a_5-a_2+\frac{1}{2}a_2a_5-\frac{1}{2}a_1a_6,$$
$$G(3)=\frac{1}{6}a_6-\frac{1}{3}a_3+\frac{1}{6}a_3a_5-\frac{1}{6}a_1\left(a_5-2a_2+a_2a_5-a_1a_6\right), \tag{34}$$
$$\ldots$$

3.2.2.2. Applying the method iterative of newton

In this part the unknown parameters are determined using the boundary conditions in Eq27. To achieve this, by defining the following residual functions, they are minimized to obtain the unknown parameters.

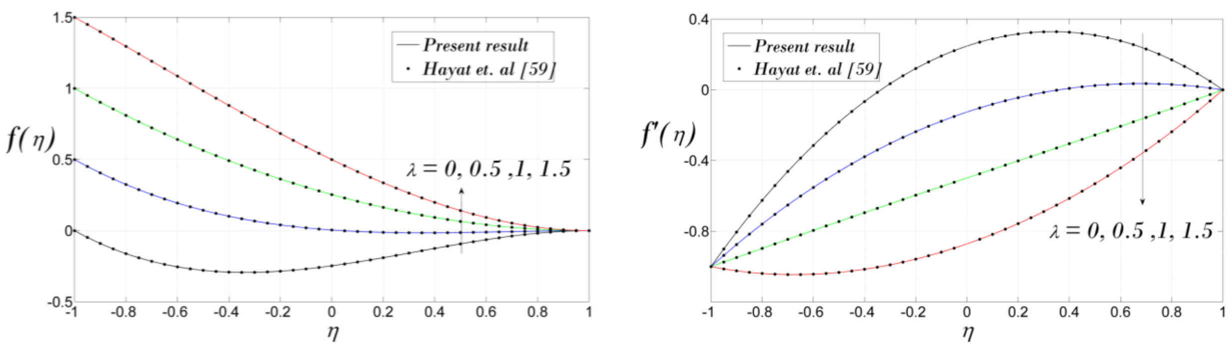
$$R_1=f\left(-1,a_1,\ldots,a_6\right)-f\left(-1\right)=\sum_{k=0}^{m}F\left(k\right)\left(-1\right)^k-\lambda,$$
$$R_2=f'\left(-1,a_1,\ldots,a_6\right)-f'\left(-1\right)=\sum_{k=1}^{m}kF\left(k\right)\left(-1\right)^{k-1}+1,$$
$$R_3=f\left(1,a_1,\ldots,a_6\right)-f\left(1\right)=\sum_{k=0}^{m}F\left(k\right),$$
$$R_4=f'\left(1,a_1,\ldots,a_6\right)-f'\left(1\right)=\sum_{k=1}^{m}kF\left(k\right),$$
$$R_5=g\left(-1,a_1,\ldots,a_6\right)-g\left(-1\right)=\sum_{k=0}^{m}G\left(k\right)\left(-1\right)^k,$$
$$R_6=g\left(1,a_1,\ldots,a_6\right)-g\left(1\right)=\sum_{k=0}^{m}G\left(k\right). \tag{35}$$

The above functions must be zero to obtain the values a_1 to a_6. To get the roots of the Eq. (35), the multi-variable iterative Newton method is used as follows:

$$\begin{bmatrix}a_1\\ \vdots\\ a_6\end{bmatrix}_{n+1}=\begin{bmatrix}a_1\\ \vdots\\ a_6\end{bmatrix}_{n}-\begin{bmatrix}\frac{\partial R_1}{\partial a_1} & \cdots & \frac{\partial R_1}{\partial a_6}\\ \vdots & \ddots & \vdots\\ \frac{\partial R_6}{\partial a_1} & \cdots & \frac{\partial R_6}{\partial a_6}\end{bmatrix}_{n}^{-1}\begin{bmatrix}R_1\\ \vdots\\ R_6\end{bmatrix}_{n},\qquad n=0,1,2,\ldots \tag{36}$$

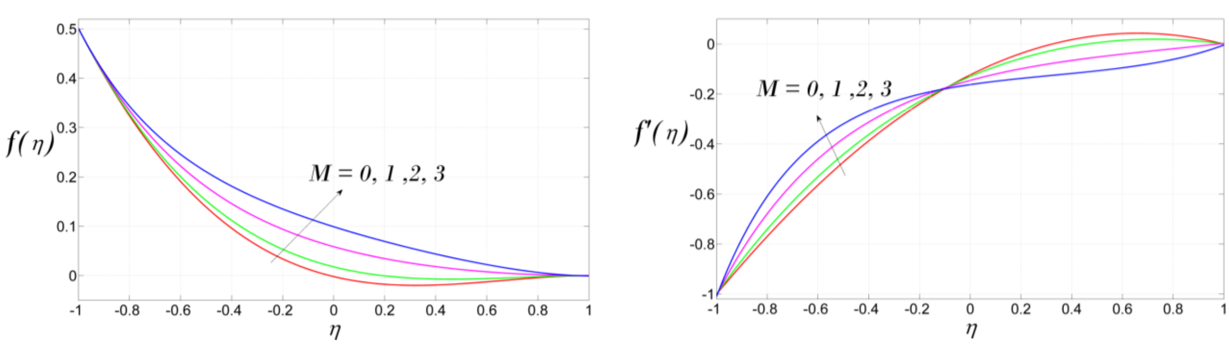
Surmising the initial values for a_1 to a_6, we must calculate the residual vector (R) and Jacobian Matrix ($\frac{\partial R_i}{\partial a_j}$). The residual vector can be determined by substituting $(a_1,...,a_6)^n$ in Eq. (35). The Jacobian matrix components in Eq36 can be calculated through differentiating them analytically in terms of a_1 to a_6 and then substituting $(a_1,...,a_6)^n$ in that equation

Fig. 4 shows the comparing the results of the present new hybrid method with HAM in [57] to show the efficiency and accuracy of it. Figs. 5 and 6 demonstrate the graphic representation of the results for different values of the parameters. The proximate solution of $f(\eta)$ and $g(\eta)$ may be seen in Table 4. The values of the unknown parameters a_1, a_2, a_3, a_4, a_5 and a_6 are shown in Table 5. These values can be substituted in the recursive relations in Eqs. (30) and (31) to be able to obtain the proximate solution of this problem.



a b

Fig. 4. Comparing the results of the proposed hybrid method with HAM [59] when $M = 0.5, K_p = 0.5$ and *Re*=0.2.



a b

Fig. 5. The profiles $f(\eta)$, $f'(\eta)$ and $g(\eta)$ when $\lambda = 0.5, K_p = 0.5$ and *Re*=0.5.

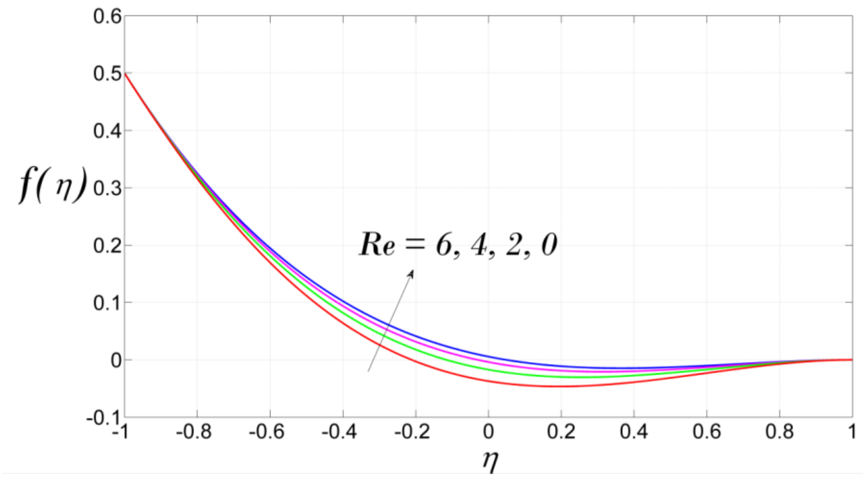


Fig. 6-a

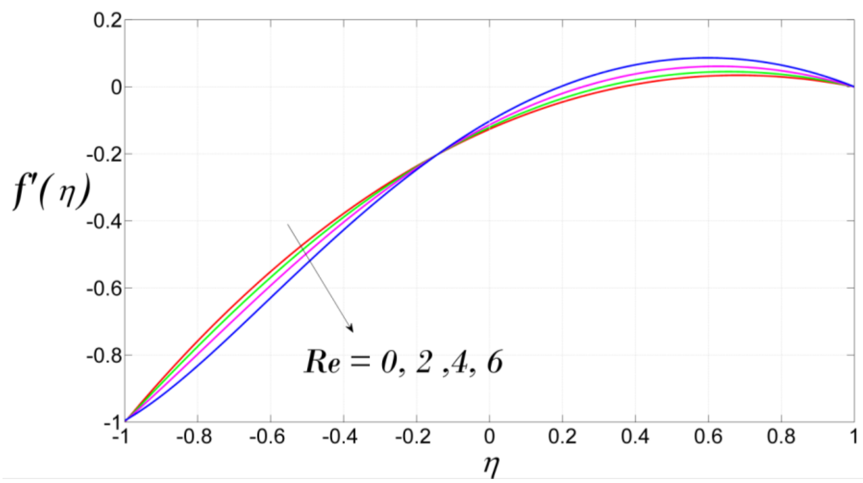


Fig. 6-b

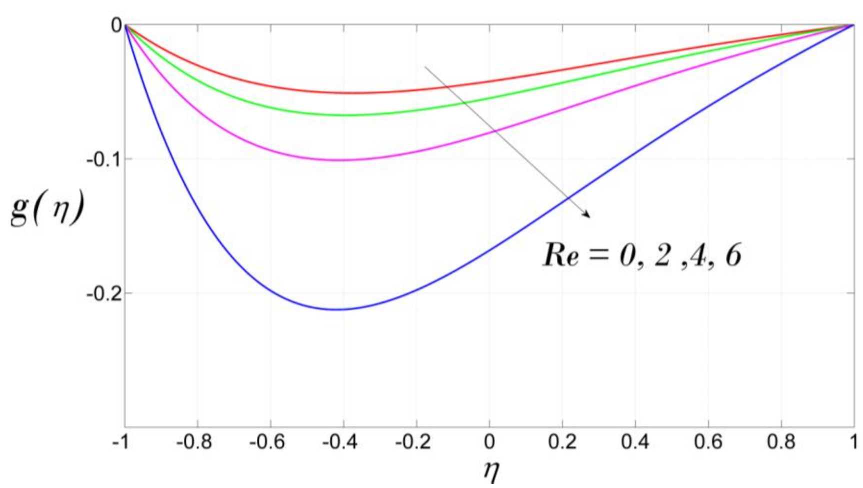


Fig. 6-c

Fig. 6. The profiles $f(\eta), f'(\eta)$ and $g(\eta)$ when $\lambda = 0.5, K_p = 0.5$ and *M*=0.5.

Table. 4
The proximate solutions $f(\eta)$ and $g(\eta)$ for K_p=0.5, M=0.5, Re=1 and different values of λ .

	Approximate solution
$\lambda = 0$	$f(\eta) = -0.2591 + 0.2503\eta + 0.2658\eta^2 - 0.2500\eta^3 - 0.0042\eta^4 - 0.0014\eta^5 + \ldots$
	$g(\eta) = 0.0228 + 0.0437\eta - 0.0512\eta^2 - 0.0449\eta^3 + 0.0279\eta^4 + 0.0013\eta^5 + \ldots$
$\lambda = 0.2$	$f(\eta) = -0.1531 + 0.1007\eta + 0.2543\eta^2 - 0.2007\eta^3 + 0.0006\eta^4 - 0.0005\eta^5 + \ldots$
	$g(\eta) = -0.0024 + 0.0424\eta - 0.0223\eta^2 - 0.0420\eta^3 + 0.0242\eta^4 - 0.0003\eta^5 + \ldots$
$\lambda = 0.4$	$f(\eta) = -0.0496 - 0.0490\eta + 0.2478\eta^2 - 0.1517\eta^3 + 0.0032\eta^4 + 0.0003\eta^5 + \ldots$
	$g(\eta) = -0.0315 + 0.0436\eta + 0.0102\eta^2 - 0.0419\eta^3 + 0.0208\eta^4 - 0.0015\eta^5 + \ldots$
$\lambda = 0.6$	$f(\eta) = 0.0514 - 0.1985\eta + 0.2462\eta^2 - 0.1029\eta^3 + 0.0034\eta^4 + 0.0011\eta^5 + \ldots$
	$g(\eta) = -0.0652 + 0.0479\eta + 0.0467\eta^2 - 0.0452\eta^3 + 0.0178\eta^4 - 0.0025\eta^5 + \ldots$
$\lambda = 0.8$	$f(\eta) = 0.1498 - 0.3476\eta + 0.2497\eta^2 - 0.0546\eta^3 + 0.0012\eta^4 + 0.0021\eta^5 + \ldots$
	$g(\eta) = -0.1044 + 0.0560\eta + 0.0878\eta^2 - 0.0524\eta^3 + 0.0158\eta^4 - 0.0034\eta^5 + \ldots$
$\lambda = 1$	$f(\eta) = 0.2455 - 0.4963\eta + 0.2585\eta^2 - 0.0073\eta^3 - 0.0034\eta^4 + 0.0034\eta^5 + \ldots$
	$g(\eta) = -0.1502 + 0.0689\eta + 0.1341\eta^2 - 0.0641\eta^3 + 0.0149\eta^4 - 0.0043\eta^5 + \ldots$

Table. 5
The values of a_1, a_2, a_3, a_4, a_5 and a_6 obtained through the iterative method of Newton when K_p=0.5, M=0.5.

		$f(0) = a_1$	$f'(0) = a_2$	$f''(0) = a_3$	$f'''(0) = a_4$	$g(0) = a_5$	$g'(0) = a_6$
	$\lambda = 0$	-0.24443	0.24657	0.47750	-1.45860	0.02838	0.03921
Re=0	$\lambda = 0.5$	0.00557	-0.12625	0.47750	-0.73480	-0.04236	0.03921
	$\lambda = 1$	0.25557	-0.49908	0.47750	-0.01099	-0.11311	0.03921
	$\lambda = 0$	-0.25911	0.25032	0.53151	-1.49862	0.02283	0.04374
Re=1	$\lambda = 0.5$	0.00121	-0.12375	0.49278	-0.76326	-0.04771	0.04531
	$\lambda = 1$	0.24547	-0.49628	0.51702	-0.04374	-0.15023	0.06886
	$\lambda = 0$	-0.32668	0.25506	0.78702	-1.55914	-0.00320	0.06262
Re=5	$\lambda = 0.5$	-0.02698	-0.10829	0.59086	-0.93862	-0.11107	0.11124
	$\lambda = 1$	0.14960	-0.45244	0.86163	-0.54167	0.39921	-0.35279

All the initial guesses for a_1 to a_6 are considered one. The convergence history of the residual and unknown parameters is presented in Fig. 7 for a special case. As it can be seen in Fig7. The problem rapidly converged with only 7 iterations since the Jacobian matrix is determined through analytically differentiating in terms of a_1 to a_6.

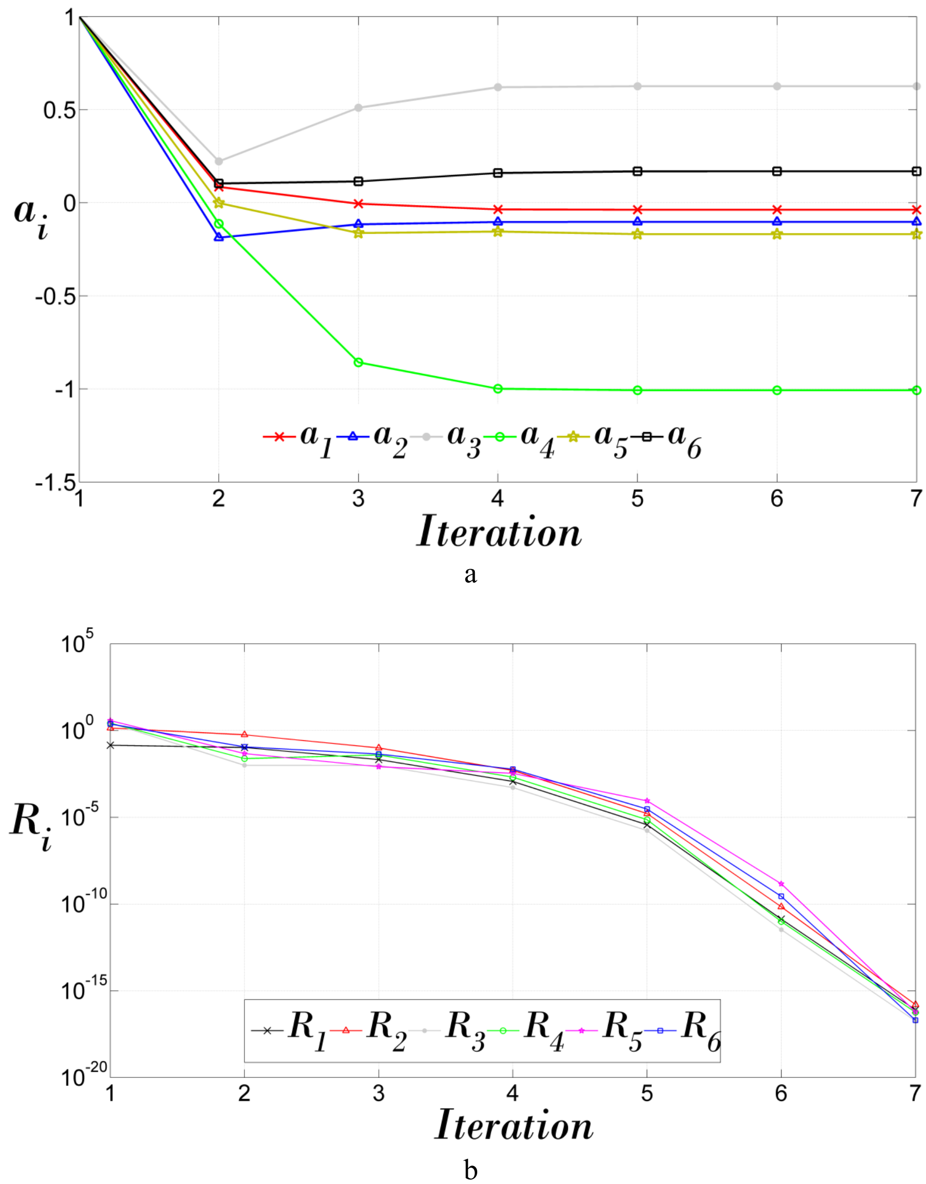


Fig. 7. The convergence history of the iterative method of Newton when K_p=0.5, M=0.5, $\lambda = 0.5$ and Re=6 for **a)** the residual functions and **b)** the unknown parameters.

4. Conclusion

The governing equations of the majority of the fluid mechanics problems can be expressed as a system of nonlinear boundary value problems. The present article introduced a new hybrid analytical-numerical procedure for solving this kind of problems The method includes the differential transform method and the iterative method of Newton. The Taylor series of the solution is computed by transforming the boundary value problems(BVP) and it s boundary conditions to a set of algebraic equations. By substituting the Jacobian matrix in the iterative method of Newton,the unknown parameters can be computed

Finally, the proximate solution of the problem is obtained in the form of a polynomial function. The application of the hybrid technique is illustrated by applying on two nonlinear problems in fluid mechanics in the literature: condensation film on the inclined rotating disk and rotating MHD flow over a porous shrinking sheet. Comparing the present results with the numerical solutions and also the results presented in the literature, an excellent accuracy is observed. The quick convergence of the solution is one of the other important merits of the method proposed.

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